Efficient methods for Parameter Identification in Differential Equation Models

New regularization functionals for sparse inverse problems

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Motivation

Many types of signals arising in nature or in technical applications can be described by a small number of significant degrees of freedom, with respect to the basis under consideration. Popular examples are images transformed to a wavelet basis or audio signals in a basis of cosine functions. To estimate such signals (see Fig. 1 for an example) out of measurements new methods have to be developed.

Sparse Inverse Problems

The solution of sparse inverse problems is a new evolving domain in the inverse problems community with a growing number of publications in the last few years. We try to solve the inverse problem:

\[ F(x) = y \]

Where,

\( F \) is an ill-posed forward operator
\( F : X \rightarrow Y \), \( X \) and \( Y \) are adequate spaces
only noisy data \( y \) is given
the solution \( x \) is sparse, i.e.

\[ x = \sum_{i=1}^{k} a_i \phi_i \]

with \( k \ll N \) and \( \{\phi_i\}_N \) the considered basis.

Bayes and Tikhonov

Based on the strong connection between Tikhonov and Bayes regularization in case of Gaussian noise and prior distribution, we searched for other sparsity enforcing distributions to gain new regularization functionals for the classical Tikhonov regularization.

\[ \pi(x|y) \propto \exp \left( -\frac{1}{2} (Ax - y)^2 + \frac{1}{2\sigma^2} \right) \]

\[ -\log(\pi(x|y)) = \frac{1}{2} (Ax - y)^2 + \frac{\sigma^2}{2\gamma^2} \]

State of current work

At the moment we work on three potential regularization functionals:

- a Cauchy regularization term
- a Betaprime regularization term
- a \( l_1 - \log \) regularization term

In Fig. 2 you see \( 60 \times 60 \) random samples from the associated prior distributions, where a black pixel means \( \approx 0 \).

Outlook

- In near future we want to show regularization properties also for the \( l_1 - \log \) term and convergence rates for the other approaches.
- Calculation of minimizers and of course other sparsity enforcing prior distributions are of interest.

Connections to SimTech Visions

- In biological systems the identification of unknown parameters is, and will in future surely enhance, a crucial task. On the way from classical to systems biology our identification methods can help to gain further and deeper insights into the differential equation systems of the future.

Cauchy

The Cauchy distribution is defined via the probability density function:

\[ p(x|\gamma) = \frac{1}{\pi \gamma} \frac{1}{1 + x^2/\gamma^2} \]

Regarding the negative logarithm, leads to:

\[ R_{\text{cauchy}}(x) = \sum_{i=1}^{N} \log(1 + x_i^2) \]

Betaprime

The Betaprime distribution is defined via the probability density function:

\[ p(x|\alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1+x)^{-\alpha-\beta} \]

for \( x \in [0,1] \). We set \( \alpha = 1 \) and replace \( x \) with \( |x| \), then the negative logarithm yields:

\[ R_{\text{betaprime}}(x) = \sum_{i=1}^{N} (1 + \beta) \log(1 + |x_i|) \]

Regularization properties

We showed for both, the Cauchy and the Betaprime regularization term, that they lead to a
- wellposed
- convergent
- stable

regularization method.

Applications

In the sparse signal recovery application the Cauchy and the \( l_1 - \log \) term lead to significantly better estimates for the approximated signal, see Fig. 5.

Another interesting application is the estimation of a sparsely distributed parameter in a PDE. For illustration purpose, we set up a simple model problem. Identify the source term \( q \) in:

\[ \Delta u = q \quad \text{in} \Omega \]

\[ u = 0 \quad \text{on} \partial \Omega \]

on the unit square \( \Omega = [0,1] \times [0,1] \subset \mathbb{R}^2 \) out of measurements \( u \), see Fig. 6.

Figure 1: Sparse Signal Recovery: The goal is to estimate the signal given to the left out of exact measurements (see the midway plot). On the right the signal recovered with Tikhonov regularization and \( l_1 \)-norm is illustrated.

Figure 2: \( 60 \times 60 \) Samples from different sparsity enforcing prior distributions (upper row, black\( \approx 0 \)). The associated density functions (lower row).

Figure 3: Cauchy regularization term with different \( \gamma \) values.

Figure 4: Betaprime regularization term with different \( \beta \) values.

Figure 5: Approximations of the signal of Fig. 1, given the same data, but with Cauchy or \( l_1 - \log \) regularization term.

Figure 6: Identified source term for the Poisson equation.