Martin boundary theory on weighted fractals

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Motivation

aim define the Laplacian on weighted fractals

reason weighted fractals exist but the previous work on Martin boundaries covers only unweighted fractals

method study behavior of harmonic functions

problem the transition probability is homogeneously defined which leads only to the unweighted case (as in figure 1)

idea modify the transition probability, such that it harmonizes with the weights

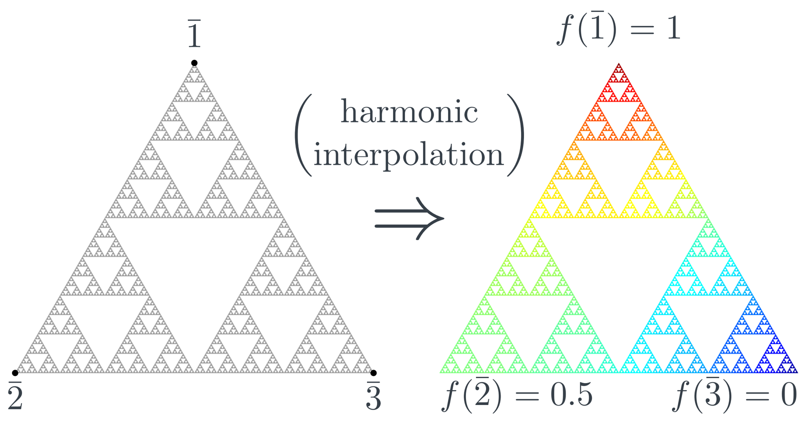

Fig. (1) harmonic function with boundary conditions on the homogeneous Sierpiński gasket

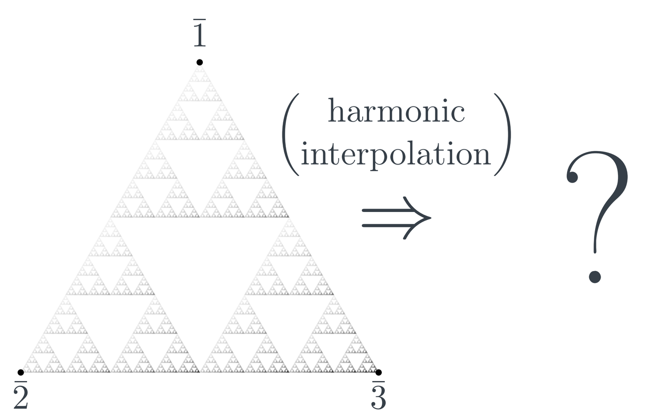

Fig. (2) open question in the weighted case

Martin boundary theory - preliminaries

Consider an IFS $\{S_1, \dots, S_N\} : D \subseteq \mathbb{R}^n \to D$ satisfying the (OSC). Assume, the attractor $K = \bigcup_{i=1}^N S_i(K)$ of the IFS is connected.
We consider the **alphabet** $\mathcal{A} = \{1, \dots, N\}$, the **word space** $\mathcal{W} := \bigcup_{n\geq 1} \mathcal{A}^n \cup \{\emptyset\}$ and denote the set of all infinite $\mathcal{A}$-valued sequences $w_1 w_2 \dots$ by $\mathcal{W}^*$.

$v, w \in \mathcal{W}$ are **equivalent** (noted by $v \sim w$), if and only if $|v| = |w|$, $S_v(K) \cap S_w(K) \neq \emptyset$ and $v^- \neq w^-$.
Set $R(w) := \#\{v \in \mathcal{W} : v \sim w\}$.
Let $p(\cdot, \cdot)$ be a **transition probability** on $\mathcal{W}$, such that

$$p(w, \tilde{w}i) := \frac{1}{R(w)N} \qquad \text{for } \tilde{w} \sim w, i \in \mathcal{A}.$$

This defines a **Markov chain** $(X_n)_{n\geq 0}$ on $\mathcal{W}$. The associated **Markov operator** P is defined by

$$(Pf)(v) := \sum_{w\in\mathcal{W}} p(v, w) f(w)$$

and we call a function f harmonic, if $Pf = f$.

The n-step transition probability from v to w is denoted by $p_n(v, w)$ with $p_0(v, w) = \delta_v(w)$. Using this, we define the **Green function** $g(v, w) : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ by

$$g(v, w) := \sum_{n=0}^{\infty} p_n(v, w)$$

where

$$g(\emptyset, w) = N^{-|w|} \qquad \text{for } w \in \mathcal{W} \tag{1}$$

holds. The **Martin kernel** $k(v, w) : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ is defined by

$$k(v, w) := \frac{g(v, w)}{g(\emptyset, w)}$$

and based on this we define the **Martin metric** ρ_M on $\mathcal{W}$ by

$$\rho_M(v, w) := \left|2^{-|v|} - 2^{-|w|}\right| + \sum_{z\in\mathcal{W}} (2N)^{-|z|}\ |k(z, v) - k(z, w)|$$

for $v, w \in \mathcal{W}$.

The **Martin space** $M = \overline{\mathcal{W}}$ is the ρ_M-completion of $\mathcal{W}$ and the **Martin boundary** ∂M is defined by $\partial M = M \backslash \mathcal{W}$. Further (M, ρ_M) is a compact metric space, so that for fixed $v \in \mathcal{W}$ every function $w \mapsto k(v, w)$ has an extension to a continuous function on M, denoted by $\xi \mapsto k(v, \xi), \xi \in M$.

In [1] Denker and Sato studied the case, that the IFS generates the $(N-1)$-dimensional Sierpiński gasket. They proved:

$$K \cong \mathcal{W}^*/_\sim \cong M$$

For this, they calculated the Martin kernel k in an explicit form.

With further assumptions on p this holds for all IFS satisfying the OSC (proved in [3] using hyperbolic boundaries).

the weighted case - results

Every $i \in \mathcal{A}$ respectively S_i gets a probability $p_i \in (0, 1)$ with $\sum_{i=1}^N p_i = 1$.
We get a mass distribution m with

$$m(w) = m(w_1 w_2 \dots w_n) = p_{w_1} p_{w_2} \cdots p_{w_n} \qquad \text{for } w = w_1 \dots w_n \in \mathcal{W}.$$

$m(w)$ can be understood as the probability getting from $\emptyset$ to w, which is known as $g(\emptyset, w)$. This contradicts equation (1) and we have to redefine p.
Consider the idea, that the probability of going from v to its child w should be equal to the quotient of the mass in w and the mass we start from, the mass of v. After scaling and some calculations we get:

$$p(v, w) := \begin{cases} \frac{m(w)}{\sum_{\hat{v}\sim v} m(\hat{v})} & \text{if } w = \hat{v}i \text{ with } \hat{v} \sim v \text{ and } i \in \mathcal{A}, \\ 0 & \text{else.} \end{cases}$$

It then holds, that $p(v, w) = p(\hat{v}, w)$ for $\hat{v} \sim v$.

Theorem 1 (K. 2018)
For all $w \in \mathcal{W}$ it holds, that $g(\emptyset, w) = m(w)$.

The calculation of $g(v, w)$ seems to be very hard, but there is a hidden structure, which we can reveal. For this define the function $q : \mathcal{W} \times \mathcal{W} \to [0, 1]$ by

$$q(v, w) := \begin{cases} \frac{g(v,w)}{m(w)} \sum_{\hat{v}\sim v} m(\hat{v}) & \text{if } v \neq w, \\ 1 & \text{if } v = w. \end{cases}$$

Lemma 2 (K. 2018)
Let $v, w \in \mathcal{W}$ and $i, j \in \mathcal{A}$. The function q fulfills then the recursive property

$$q(v, wi) = \begin{cases} 1 & \text{if } w \sim v, \\ \frac{\sum_{\widetilde{w}\sim w} q(v,\widetilde{w}) m(\widetilde{w})}{\sum_{\widetilde{w}\sim w} m(\widetilde{w})} & \text{if } w \not\sim v. \end{cases}$$

Theorem 3 (K. 2018)
Let $v, w, \widetilde{w} \in \mathcal{W}$ and $\widetilde{w} \sim w$. If $w^- \sim (\widetilde{w})^-$ holds, then $q(v, w) = q(v, \widetilde{w})$ follows.

Theorem 4 (K. 2018)
If for all $w \in \mathcal{W}$ holds either

$$m(w) = m(\widetilde{w})\ \ \forall \widetilde{w} \sim w \qquad \text{or} \qquad w^- \sim (\widetilde{w})^-\ \ \forall \widetilde{w} \sim w$$

then

$$q(v, wi) = \frac{1}{R(w)} \sum_{\widetilde{w}\sim w} q(v, \widetilde{w})$$

*holds for all $v \in \mathcal{W}$ with $w \not\sim v$. In particular is q **independent** from m.*

Lemma 5 (K. 2018)
Let $v, w, \widetilde{w} \in \mathcal{W}$ with $\widetilde{w} \sim w$. If $w^- \sim (\widetilde{w})^-$ and $w \not\sim v$ holds, then $k(v, w) = k(v, \widetilde{w})$ holds.

Theorem 6 (K. 2018)
If the homogeneous Martin kernel k_{hom} can be computed and theorem 4 holds, then it follows, that for $v, w \in \mathcal{W}$ the Martin kernel can be calculated by:

$$k(v, w) = \begin{cases} k_{hom}(v, w) \frac{R(v) \cdot N^{-|v|}}{\sum_{\hat{v}\sim v} m(\hat{v})} & \text{for } v \neq w, \\ \frac{1}{m(v)} & \text{for } v = w. \end{cases}$$

Thus we can calculate the Martin kernel in the weighted case.

Open questions and problems

❶ Under which conditions can theorem 6 be generalized?
❷ For which fractal/IFS (besides the Sierpiński gasket) are the preconditions from theorem 6 satisfied?

References

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